Mechanics

Simplest growth model is something like

$$y(t) = y_0 e^{gt} \tag{1}$$

- $y(t)$ is output per worker at time t
- y_0 is output per worker at some initial time 0
- g is the growth rate of output per worker
- e^{gt} gives us exponential growth

Simpler to draw/graph/analyze this in log terms, so take

$$\ln y(t) = \ln y_0 + gt. \tag{2}$$

Rules for Logs

I used a few rules for taking logs there. Here are the ones you need to know.

- $\ln XZ = \ln X + \ln Z$
- $\ln X/Z = \ln X - \ln Z$
- $\ln X^{\alpha} = \alpha \ln X$
- $\ln e^{X} = X$

"take logs" and "take the natural log" are synonyms in this class. We are always implicitly using natural logs - the $\ln X$ symbol.

These rules roll up on one another, so

$$\ln X^{\alpha} Z/W^{\beta} = \alpha \ln X + \ln Z - \beta \ln W \tag{3}$$

Log Output per worker

Our simple model

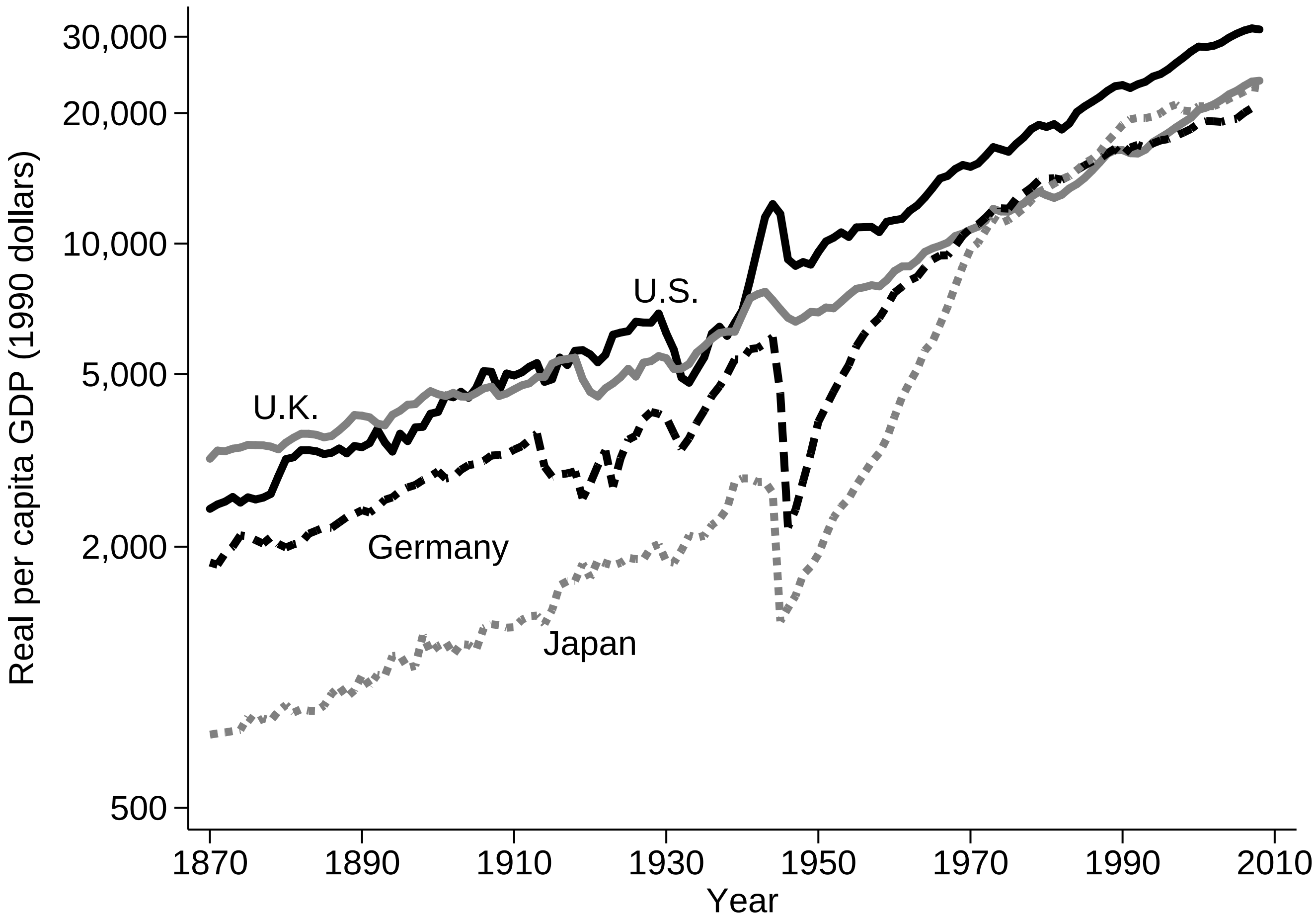
$$\ln y(t) = \ln y_0 + gt. \tag{4}$$

describes a line. $\ln y(t)$ is the y variable, t is the x.

- y_0 is the intercept
- g is the slope

Log Output per worker

$y(t)$ is roughly linear for lots of advanced countries



Identifying Different Effects

In terminology, distinguish

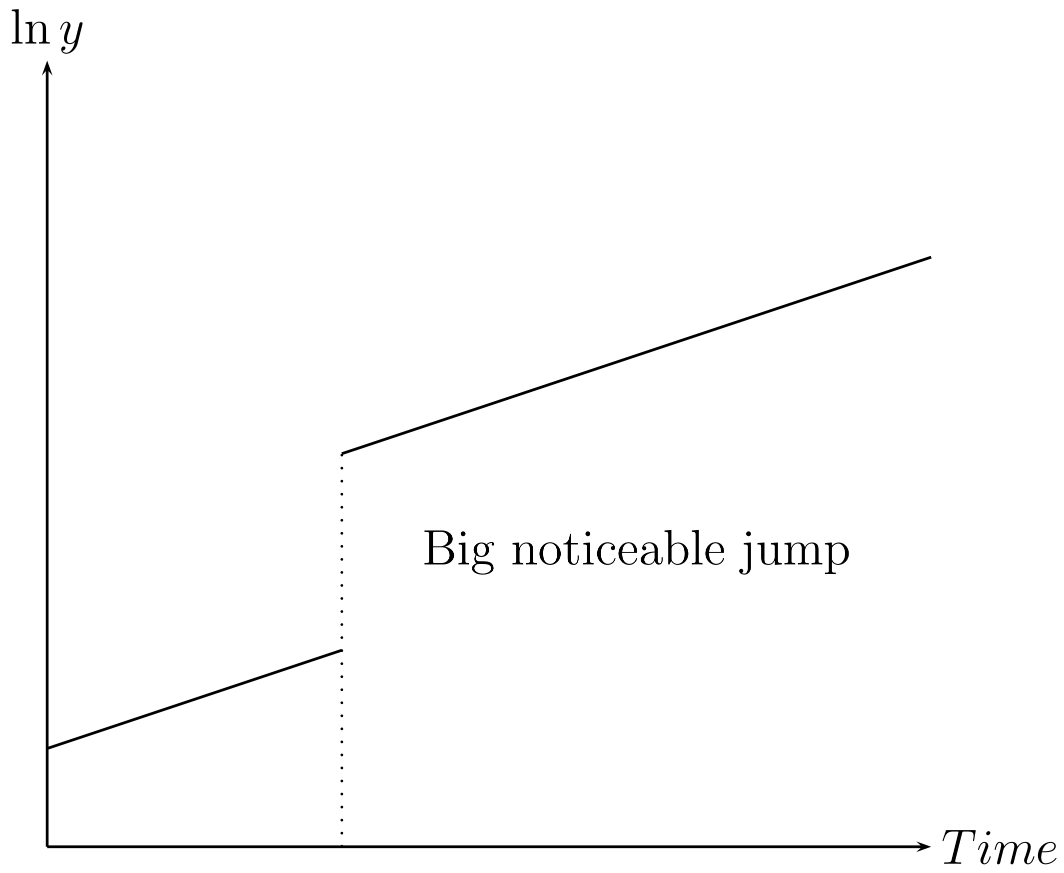
- *Level Effects*: These are things that shift the intercept, y_0, up or down. In the early 1900's Japan was at a distinctly lower *level* than the U.S. or U.K.
- *Growth Effects*: These are things that shift the slope, g. For the most part, g is constant for all those countries in the diagram.
- *Transitional Effects*: This is growth that occurs as a country moves from one level to another. Temporary spurt in growth rate (a very steep line) that catches a country up to some level. See Japan and Germany in the diagram after WWII.

As a preview, here is what we are going to see in the class

- The growth rate, g, is probably the same across countries, or very close. There are very few examples of true growth effects.
- Much of the distinction in observed growth rates between countries is due to transitional growth - China catching up to the West, for example.
- Much of the explanation for why some countries are poor and some are rich is due to level effects.

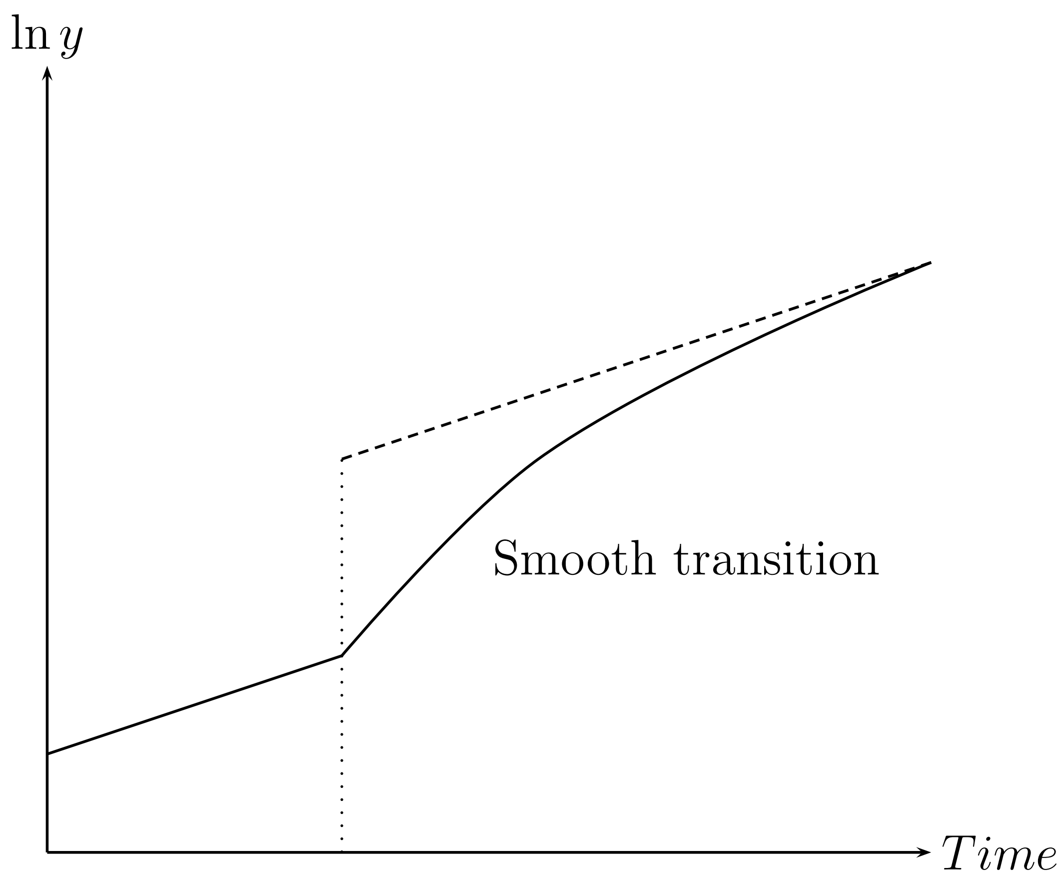
Transitional and Level Effects

Our simple model is too simple to capture transitional effects, which occurs because output per worker doesn't have to be exactly on the line described by $y(t) = y_0 e^{gt}$. If y_0 jumps at some point, then our simple model predicts:



Transitional and Level Effects

But we don't see leaps like this in the data. We see



Note that for a while output is not on the line described by our simple model.

Transitional and Level Effects

The Solow Model we'll start with captures this smooth transition

- Output depends on stock of accumulated capital (physical machines, human capital, etc..)
- Output cannot "jump" to new level even if y_0 jumps
- Takes time to accumulate capital to reach new level

A level effect (jump in y_0) will cause transitional effects - Japan after WWII

Or, if y falls "off" the simple line, we'll get transitional growth - Germany after WWII

Calculating Growth Rates

We would typically write

$$\frac{y(t+k)-y(t)}{y(t)} \tag{5}$$

to calculate the percent growth rate from period t to period $t+1$. Given our simple model, this is

$$\frac{y(t+1)-y(t)}{y(t)} = \frac{y(t+1)}{y(t)} - 1 = \frac{y_0 e^{g(t+1)}}{y_0 e^{gt}} - 1 = e^g - 1 \tag{6}$$

It is true that $e^g \approx 1+g$ for small values of g (like $g < 0.05$). So therefore

$$\frac{y(t+1)-y(t)}{y(t)} \approx g \tag{7}$$

Or, take the difference in log output

$$\ln y(t+1) - \ln y(t) = \ln y_0 + g(t+1) - \ln y_0 - gt = g \tag{8}$$

So we will typically take the difference in logs to find the growth rate. This is essentially the same as calculating the percent change like we did above.

Logs and Time Derivatives

We'll be using this a lot. It's a mechanical technique to find out how some function grows over time. In our example, we've got $y(t) = y_0 e^{gt}$. We've already taken logs

$$\ln y(t) = \ln y_0 + gt \tag{9}$$

and now we need to take the time derivative

$$\frac{\partial \ln y(t)}{\partial t} = \frac{\partial \ln y_0}{\partial t} + \frac{\partial gt}{\partial t} \tag{10}$$

and this is

$$\frac{\partial y(t) \partial t}{y(t)} = g \tag{11}$$

where I've assumed that y_0 is constant over time.

For notational convenience, write

$$\frac{\partial y(t)/\partial t}{y(t)} = \frac{\dot{y}}{y} = g \tag{12}$$

where the $\dot{y}$ just means the change in y over time.

Note that "take logs and derivatives" is really just like looking at a percent change (difference in y over y).

Logs and Time Derivatives

With production functions, we'll be doing all sorts of log-ing and derivative-ing. Take

$$Y = K^{\alpha} X^{\beta} L^{1-\alpha-\beta} \tag{13}$$

as an example. Taking logs we get

$$\ln Y = \alpha \ln K + \beta \ln X + (1 - \alpha - \beta) \ln L \tag{14}$$

using our rules for logs. Then time derivatives are

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + \beta \frac{\dot{X}}{X} + (1 - \alpha - \beta) \frac{\dot{L}}{L} \tag{15}$$

This tells us that the growth rate on the left (of Y) is equal to a sum of the growth rates of the things on the right.

Steady State

A *steady state* is a situation in which a dynamic variable stops changing. That is, perhaps we have some variable K described by

$$\dot{K} = sK^{\alpha} - \delta K \tag{16}$$

which could also be divided by K to show things in terms of growth rates

$$\frac{\dot{K}}{K} = sK^{\alpha-1} - \delta. \tag{17}$$

Regardless, where will K have a steady state? That is, at what value of K will K stop changing? $\dot{K}$ measures the change in K, so it is where $\dot{K} = 0$, or where

$$sK^{\alpha} = \delta K \tag{18}$$

which implies that

$$K^{*} = \left(\frac{s}{\delta}\right)^{1/(1-\alpha)} \tag{19}$$

where K^{*} is used to denote this steady state. If $K = K^{ast}$, then K will not change, and therefore will remain *steady* at the value K^{*} forever.

Real GDP Comparisons

We talk about $y(t)$ as output per worker. How do we measure that? In particular, how do we compare that across countries, who use different currencies and have different relative prices?

- Convert GDP in each country to common currency (dollar or "international dollar") using exchange rates
- Find real output of each "good" by dividing domestic expenditure by domestic price
- Calculate "Real GDP" by valuing real output of each good at a given international price

Example. India spends \$3,000 per capita on food and \$1,000 per capita on cell phones. Indian prices are \$2 per unit of food and \$10 per phone. Intl. price is \$3 per unit of food and \$5 per phone

- Domestic GDP is $3000 + 1000 = 4000$.
- Real output is $3000/2 = 1500$ units of food and $1000/10 = 100$ units of cell phones.
- Real GDP is $1500 \times 3 + 100 \times 5 = 5000$

Real GDP Comparisons

The set of prices we use to value goods matter a lot in international comparisons.

- International Comparison of Prices project collects prices for goods/services across the globe
- Comes up with "common price" for each good - but this is weighted heavily towards rich countries
- So real GDP measures we use in class value each country's output at rich country prices
- This can have distorting effects - in general it makes poor countries look relatively well off
- Why? Because the common price is high for things that poor countries have a lot of (services) but low for things that poor countries have little of (high-tech goods)

There is no "right" set of prices to use to compare countries. But you have to pick one set of prices to use for comparison.